

RANDOM DIRICHLET SERIES ARISING FROM RECORDS

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ABSTRACT. We study the distributions of the random Dirichlet series with parameters (s, β) defined by

$$S = \sum_{n=1}^{\infty} \frac{I_n}{n^s},$$

where (I_n) is a sequence of independent Bernoulli random variables, I_n taking value 1 with probability $1/n^\beta$ and value 0 otherwise. Random series of this type are motivated by the record indicator sequences which have been studied in extreme value theory in statistics. We show that when $s > 0$ and $0 < \beta \leq 1$ with $s + \beta > 1$ the distribution of S has a density; otherwise it is purely atomic or not defined because of divergence. In particular, in the case when $s > 0$ and $\beta = 1$, we prove that for every $0 < s < 1$ the density is bounded and continuous, whereas for every $s > 1$ it is unbounded. In the case when $s > 0$ and $0 < \beta < 1$ with $s + \beta > 1$, the density is smooth. To show the absolute continuity, we obtain estimates of the Fourier transforms, employing van der Corput's method to deal with number-theoretic problems. We also give further regularity results of the densities, and present an example of non-atomic singular distribution which is induced by the series restricted to the primes.

1. INTRODUCTION

The purpose of this paper is to analyse a class of probability distributions defined by infinite series of the following type: Let I_1, I_2, \dots be a sequence of independent random variables I_n taking values 0 or 1 with $\mathbb{P}(I_n = 1) = 1/n$. Define a random series by

$$S_o := \sum_{n=1}^{\infty} \frac{I_n}{n}. \quad (1)$$

Note that the series converges almost surely since its expectation is finite, and its distribution has the support $[1, \infty)$. A central question we consider is whether this distribution has a density or not. We show this distribution does have a density, but leave open the questions of whether this density is bounded, or continuous.

This distribution arises from the study of records in statistics. Let U_1, U_2, \dots be a sequence of independent uniform $[0, 1]$ variables, and let I_1, I_2, \dots be the associated sequence of *record indicators*:

$$I_n := 1(U_n > U_j \text{ for all } 1 \leq j < n) \quad (2)$$

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meaning that $I_n = 1$ if U_n exceeds all previous values, and $I_n = 0$ otherwise. Rényi [Rén] showed that the record indicators are independent with $\mathbb{P}(I_n = 1) = 1/n$ for all $n \geq 1$. Related properties of the record indicator sequence and its partial sums, counting numbers of records, have been extensively studied. See e.g. the monographs of Arnold et al. [ABN], and Nevzorov [Nev]. See also [Pit, Chapter 3] for related topics and references therein. We show in Theorem 2.1 that the conditional expectation of U_1 given I_1, I_2, \dots is

$$\mathbb{E}[U_1 | I_1, I_2, \dots] = \prod_{n=2}^{\infty} \left(1 - \frac{I_n}{n}\right). \quad (3)$$

In this setting, the random series (1) approximates the logarithm of (3).

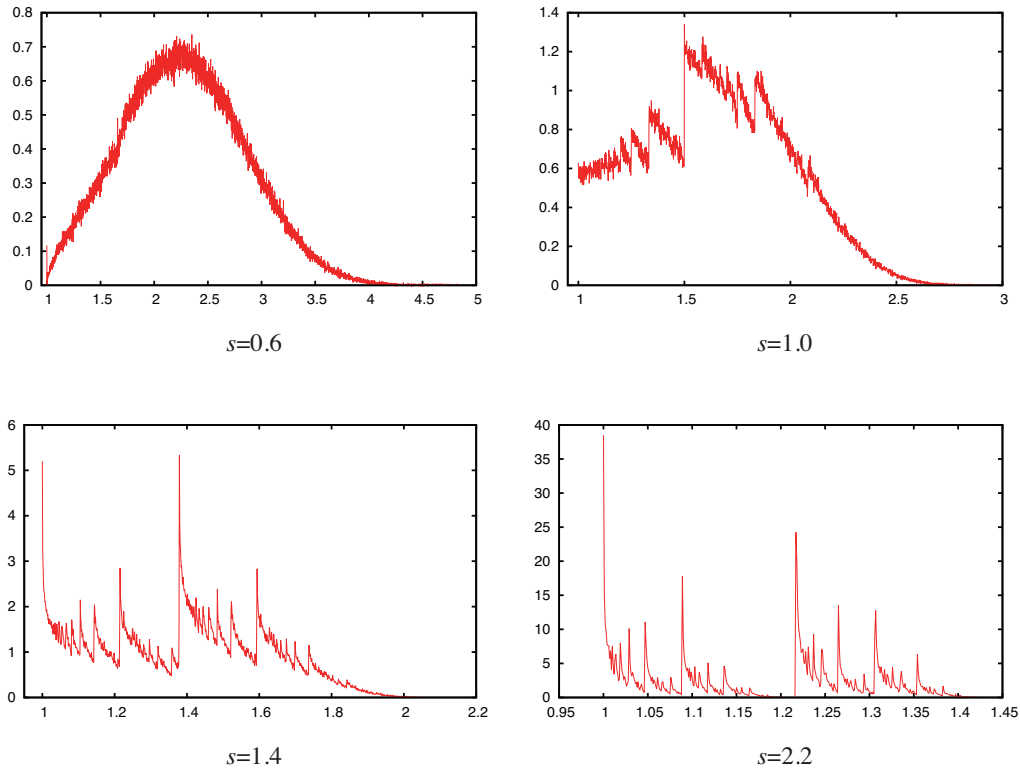


FIGURE 1. Simulation approximations to the densities of $S := \sum_{n=1}^N I_n/n^s$ ($I_n = 1$ with probability $1/n$ and 0 otherwise) for $s = 0.6, 1.0, 1.4$ and 2.2 , where $N = 10^4$, the number of samples is 10^6 and the bin size is 10^{-3} .

From the viewpoint of the study of random series, it is natural to parameterize the series (1) as follows: For $s > 0$, let

$$S := \sum_{n=1}^{\infty} \frac{I_n}{n^s}, \quad (4)$$

where I_1, I_2, \dots is a sequence of independent random variables I_n with values 0 or 1 with probability $1 - 1/n^\beta$ or $1/n^\beta$, respectively, with β a positive parameter. The series converges almost surely if $s + \beta > 1$, and just diverges almost surely if $s + \beta \leq 1$ by Kolmogorov's three-series theorem [Dur, Theorem 2.5.4]. We recover (1) when $s = 1$ and $\beta = 1$. Let μ be the distribution of S defined by (4). Jessen and Wintner showed that every convergent infinite convolution of discrete measures is of *pure type*: it is either atomic singular (purely discontinuous), non-atomic singular (continuous singular), or absolutely continuous with respect to Lebesgue measure [JW]. We observe that if $\beta > 1$, then the sequence (I_n) consists of only finitely many ones almost surely, hence μ is atomic singular, and in fact, it is supported on the countable set of all possible values of the finite sums $\sum \varepsilon_n/n^s$, where ε_n is 0 or 1.

Theorem 1.1. *Let $s > 0$ and $0 < \beta \leq 1$ with $s + \beta > 1$.*

- (1) *For $\beta = 1$, the distribution μ is absolutely continuous with respect to Lebesgue measure for all $s > 0$. For every $0 < s < 1$, the distribution μ has a bounded continuous density, whereas for every $s > 1$, it has an unbounded density. Moreover, for each $s > 0$ the Fourier transform $\hat{\mu}$ of μ has the following property: for every small enough $\varepsilon > 0$ there exists a constant $C_{\varepsilon,s} > 0$ such that for all real t ,*

$$|\hat{\mu}(t)| \leq C_{\varepsilon,s} |t|^{-\frac{1}{s} + \varepsilon}. \quad (5)$$

- (2) *For $0 < \beta < 1$, for every $s > 1 - \beta$, the distribution μ has a smooth density. Moreover, there exist constants $C_{\beta,s} > 0$ and $T > 0$ such that for all real t with $|t| \geq T$,*

$$|\hat{\mu}(t)| \leq \exp \left(-C_{\beta,s} |t|^{\frac{1-\beta}{s+1}} \right).$$

In Section 3 and Section 4, we prove this theorem; see Theorem 3.1, Theorem 4.1 and Theorem 4.2. In part 1 of Theorem 1.1, we observe a transition of boundedness of the densities on the line $\beta = 1$ at $s = 1$. This leaves open questions about the density in the critical case when $\beta = 1$ and $s = 1$, which we present in Section 6. Figure 1 displays the densities of μ for $\beta = 1$ with several values of s .

Random series have been studied in various contexts. If one considers the random harmonic series $\sum_{n=1}^{\infty} \pm 1/n$, where the signs are chosen independently with equal probability, then one can show that the distribution has a smooth density [Sch]. In the context of random functions, the random series $\sum_{n=1}^{\infty} \pm 1/n^s$ is called a *random Dirichlet series* and has been studied mainly by focusing on its analytic properties (e.g., [BM] and references therein). In most cases, however, it is assumed that the random signs or the random coefficients are independent identically distributed or satisfy certain uniformity. The random Dirichlet series with coefficients I_n as in (4) has not been extensively studied yet. The random Dirichlet series (4) can also be compared with the random geometric series $\sum_{n=1}^{\infty} \pm \lambda^n$, where the signs are chosen independently with equal probability and λ is a parameter between 0 and 1. This distribution has been studied under the name of Bernoulli convolutions, and its absolute continuity/singularity problem has attracted a lot of attention. See the expository article by Peres, Schlag and Solomyak [PSS] and recent notable progress by Hochman [H] and by Shmerkin [Shm].

Let us show further regularity results of the densities of μ . The following is an implication of the decay of the Fourier transform in the case when $0 < s < 2$ and $\beta = 1$.

Corollary 1.2. *Let $s > 0$ and $\beta = 1$. For an integer $r \geq 0$ and for every $0 < s < 1/(r+1)$, the distribution μ has a density in C^r , and for every $1 \leq s < 2$, it has a density in L^q for every $1 \leq q < s/(s-1)$, where $s/(s-1) = \infty$ when $s = 1$.*

Let us mention another implication of the decay of the Fourier transform for $0 < s < 2$ and $\beta = 1$. The fractional derivatives are expressed in terms of the $(2, \gamma)$ -Sobolev space L^2_γ , where the norm is defined by $\|\mu\|_{2,\gamma}^2 = \int_{-\infty}^{\infty} |\hat{\mu}(t)|^2 |t|^{2\gamma} dt$. Finiteness of $\|\mu\|_{2,\gamma}$ for some positive $\gamma > 0$, implies that μ has γ -fractional derivatives in L^2 . The following is an immediate consequence of part 1 of Theorem 1.1.

Corollary 1.3. *Suppose that $0 < s < 2$ and $\beta = 1$. For every $0 < \gamma < \frac{1}{s} - \frac{1}{2}$, the density of μ has γ -fractional derivatives in L^2 .*

In the proof of Theorem 1.1, we require an estimate of exponential sums by Weyl and van der Corput ([GK], [KN]) to bound the Fourier transform of the distribution μ . In the cases when $0 < s < 2$ and $\beta = 1$, and when $s > 0$ and $0 < \beta < 1$ with $s + \beta > 1$, respectively, the decay of the Fourier transform suffices to conclude that the distribution μ has a density, since $\hat{\mu}$ is in L^2 in these parameter sets, and further regularity results also follow from these estimates. In the case when $s \geq 2$ and $\beta = 1$, however, the absolute continuity of the distribution μ does not follow directly from the estimate of the Fourier transform (5) in part 1 of Theorem 1.1. In Section 4, we show that μ is absolutely continuous for all $s > 0$ and $\beta = 1$, employing a conditioning argument combined with van der Corput's method. We remark that the method which we use there does not yield further regularity of the densities for $s \geq 2$ and $\beta = 1$ unlike the case for $0 < s < 2$ and $\beta = 1$. For the sharpness of the estimate of the Fourier transform for $0 < s < 2$ and $\beta = 1$, see Remark 4.3.

To summarize the results, for (s, β) with $s + \beta > 1$, the distribution μ of (4) is always absolutely continuous except for the trivial case when $\beta > 1$.

We present one more result which provides a non-atomic singular distribution, restricting to the prime numbers sequence: Let p be a prime. Consider an independent sequence (I_p) with value 0 or 1 with probability $1 - 1/p$ or $1/p$, respectively, and the following random series:

$$S_{primes} := \sum \frac{I_p}{p^s},$$

where the summation runs over all primes p and $s > 0$. Notice again that S_{primes} is finite almost surely.

Theorem 1.4. *For every $s > 0$, the distribution μ of S_{primes} is non-atomic singular.*

Erdős proved that the asymptotic distribution of the additive function $f_n = \sum 1/p^s$, where the summation runs over all prime divisors of n is non-atomic singular [Erd]. The proof of Theorem 1.4 uses essentially the same method as Erdős'.

The organisation of this paper is as follows: In Section 2, we discuss the record indicator and the background, and then obtain the formula for the conditional expectation (3) in Theorem 2.1. In Section 3, we show a part of Theorem 1.1 concerning the estimate of the Fourier transform in Theorem 3.1 and deduce Corollary 1.2. In Section 4, we prove the absolute continuity of the distribution μ for all $s > 0$ and $\beta = 1$ in Theorem 4.1, and the unboundedness of the densities for all $s > 1$ and $\beta = 1$ in Theorem 4.2. Then we complete the proof of Theorem 1.1. In Section 5, we prove Theorem 1.4. In Section 6, we discuss the boundedness of the density at the critical case when $s = 1$ and $\beta = 1$, and present some open problems.

Notation 1.5. Throughout this article, we use c, C, c_1, c_2, \dots , to denote absolute constants whose exact values may change from line to line, and also use them with subscripts, for instance, C_ε to specify its dependence only on ε . For functions f and g , we write $f \asymp g$ if there exist some absolute constants $c_1, c_2 > 0$ such that $c_1 g \leq f \leq c_2 g$.

2. RECORDS AND PROBABILISTIC MOTIVATIONS

We start with the following question about the sequence of record indicators I_1, I_2, \dots derived from independent uniform $[0, 1]$ variables U_1, U_2, \dots as in (2). How much information does the sequence of record indicators reveal about the value of U_1 ? The answer, provided below, may be compared with the answer to the corresponding question if (I_1, I_2, \dots) is replaced by (X_1, X_2, \dots) where $X_n := 1(U_n < U_1)$. In this case, U_1 is recovered from (X_1, X_2, \dots) with probability one as the almost sure limit of S_n/n as $n \rightarrow \infty$, where $S_n := \sum_{k=1}^n X_k$ is the number of ones in the first n places. For the record indicators (I_1, I_2, \dots) , the value U_1 cannot be fully recovered from the record indicators (I_1, I_2, \dots) . More precisely, we establish the following theorem:

Theorem 2.1. *Let $V_\infty := \mathbb{E}[U_1 | I_1, I_2, \dots]$. Then*

$$V_\infty = \prod_{n=2}^{\infty} \left(1 - \frac{I_n}{n}\right).$$

Moreover,

$$\mathbb{E}V_\infty^2 = \frac{-\cos(\sqrt{5}\pi/2)}{\pi} = 0.29667513474359 \dots < \frac{1}{3} = \mathbb{E}(U_1^2). \quad (6)$$

Since

$$\mathbb{E}(U_1 - V_\infty)^2 = \mathbb{E}(U_1^2) - \mathbb{E}(V_\infty^2) > 0$$

the difference between $\mathbb{E}V_\infty^2$ and $\mathbb{E}U_1^2$ reflects the fact that U_1 is not a measurable function of all the record indicators (I_1, I_2, \dots) . The Jessen-Wintner law of pure types implies that the distribution of the infinite product V_∞ is either singular, or absolutely continuous with respect to Lebesgue measure [JW]. Define a random series by taking the logarithm and the positive sign:

$$S_\infty := -\log V_\infty = -\sum_{n=2}^{\infty} \log \left(1 - \frac{I_n}{n}\right).$$

Since $-\log(1 - I_n/n) \asymp I_n/n$, one can expect that the above sum is approximated by S_o in (1). Actually, the following holds in the same way as in the case of S_o .

Theorem 2.2. *Let μ_∞ be the distribution of S_∞ . Then μ_∞ has a density in L^q for every $1 \leq q < \infty$. In particular, the distribution of $V_\infty = \mathbb{E}[U_1|I_1, I_2, \dots]$ is absolutely continuous with respect to Lebesgue measure.*

We prove Theorem 2.2 at the end of Section 3 as a consequence of more general facts. In this section we establish Theorem 2.1. First, we obtain the conditional distribution of U_1 given the record indicators (I_1, I_2, \dots) . We denote by $\text{beta}(a, b)$ for $a, b > 0$ the probability distribution on $[0, 1]$ whose density at $u \in (0, 1)$ relative to length measure is proportional to $u^{a-1}(1-u)^{b-1}$.

Theorem 2.3. *For $n \geq 1$, let*

$$M_n := \max_{1 \leq i \leq n} U_i.$$

Then

$$U_1 = M_n \prod_{j=2}^n \frac{M_{j-1}}{M_j} \quad (7)$$

where the random variables M_n and M_{j-1}/M_j , $2 \leq j \leq n$ are independent, with M_n distributed $\text{beta}(n, 1)$ and M_{j-1}/M_j distributed as a mixture with weights $1 - 1/j$ and $1/j$ of a point mass at 1 and a $\text{beta}(j-1, 1)$ distribution on $(0, 1)$.

The conditional distribution of U_1 given I_1, \dots, I_n is described by (7) where given I_1, \dots, I_n the M_n and M_{j-1}/M_j for $2 \leq j \leq n$ are conditionally independent, with

- (i) $M_{j-1}/M_j = 1$ if and only if $I_j = 0$,
- (ii) M_n distributed $\text{beta}(n, 1)$,
- (iii) M_{j-1}/M_j distributed $\text{beta}(j-1, 1)$ if $I_j = 1$.

Proof. Note that M_n is distributed $\text{beta}(n, 1)$ since $\mathbb{P}(M_n < x) = \mathbb{P}(U_1 < x)^n$, and the ratio M_{j-1}/M_j is distributed as indicated since $\mathbb{P}(M_{j-1}/M_j = 1) = \mathbb{P}(I_j = 0) = 1 - 1/j$, and for $x < 1$, $\mathbb{P}(M_{j-1}/M_j < x) = \mathbb{P}(M_{j-1} < xU_j) = \int_0^1 (xu)^{j-1} du$. The asserted joint distribution of the n factors in (7) is established by induction on n , using $M_{n+1} = \max(M_n, U_{n+1})$, where U_{n+1} is independent of M_n and M_{j-1}/M_j , $2 \leq j \leq n$. Indeed, it is enough to show that M_n and M_{n-1}/M_n are independent and this can be checked directly.

It is clear by definition of I_j that (i) above holds. So the I_j are functions of the independent ratios M_{j-1}/M_j , hence independent as j varies with $\mathbb{P}(I_j = 1) = 1/j$, as found by Rényi [Rén]. The independence of M_n and the I_j for $1 \leq j \leq n$ is well-known [Nev, Lemma 13.2]. Thus, given I_1, \dots, I_n , the M_n and the ratios M_{j-1}/M_j for $2 \leq j \leq n$ are conditionally independent. It follows easily that the conditional distribution of M_n and M_{j-1}/M_j given I_1, \dots, I_n is as indicated in (ii) and (iii). \square

Now we prove Theorem 2.1:

Proof of Theorem 2.1. Since the mean of $\text{beta}(a, b)$ is $a/(a + b)$, we read from Theorem 2.3 that

$$V_n := \mathbb{E}(U_1 | I_1, \dots, I_n) = \frac{n}{n+1} \prod_{j=2}^n \left(1 - \frac{I_j}{j}\right). \quad (8)$$

By the bounded martingale convergence theorem,

$$V_\infty := \mathbb{E}(U_1 | I_1, I_2, \dots) = \prod_{j=2}^{\infty} \left(1 - \frac{I_j}{j}\right). \quad (9)$$

Note in passing that the limiting infinite products considered here exist not only almost surely, as guaranteed by martingale convergence, but in fact for all sequences of 0/1 values of I_2, I_3, \dots , allowing 0 as a possible limit. This is obvious by inspection of the infinite products, since the partial products are non-increasing.

The mean square of V_∞ is given by

$$\mathbb{E}(V_\infty^2) = \prod_{j=2}^{\infty} \left(1 - \frac{1}{j} + \frac{1}{j} \left(1 - \frac{1}{j}\right)^2\right). \quad (10)$$

The j -th factor in (10) is

$$1 - \frac{1}{j} + \frac{1}{j} \left(1 - \frac{1}{j}\right)^2 = \frac{(j-1)(j^2 + j - 1)}{j^3} \quad (11)$$

$$= \frac{(j-1)}{j^4} ((j-1)(j+1)^2 + 1) \quad (12)$$

$$= \left(\frac{j-1}{j}\right)^2 \left(\frac{j+1}{j}\right)^2 \left(1 + \frac{1}{(j-1)(j+1)^2}\right) \quad (13)$$

There is some telescoping of the product, with the simplification

$$\mathbb{E}(V_\infty^2) = \frac{1}{4} \prod_{j=2}^{\infty} \left(1 + \frac{1}{(j-1)(j+1)^2}\right)$$

with the finite n version, using (8),

$$\mathbb{E}(V_n^2) = \frac{1}{4} \prod_{j=2}^n \left(1 + \frac{1}{(j-1)(j+1)^2}\right), \quad (14)$$

which increases to its limit as n increases.

To prove the formula (6), we show first that the finite product in (14) can be evaluated as

$$\mathbb{E}(V_n^2) = \frac{5n\Gamma\left(n + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)\Gamma\left(n + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)}{(n+1)!^2\Gamma\left(\frac{7}{2} - \frac{\sqrt{5}}{2}\right)\Gamma\left(\frac{7}{2} + \frac{\sqrt{5}}{2}\right)}. \quad (15)$$

Indeed, this formula holds for $n = 1$, with both sides equal to $1/4$, by interpreting the empty product in (14) as 1, and using the gamma recursion $\Gamma(r+1) = r\Gamma(r)$.

The proof for general n is by induction. Assuming that (15) has been established for n , the formula with $n + 1$ instead of n is deduced from the identity

$$1 + \frac{1}{n(n+2)^2} = \frac{(n+1)(n^2+3n+1)}{n(n+2)^2},$$

by using the gamma recursion to expand

$$n^2 + 3n + 1 = \left(n + \frac{3}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 = \frac{\Gamma\left(n+1+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\Gamma\left(n+1+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)}{\Gamma\left(n+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\Gamma\left(n+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)}.$$

Euler's reflection formula for the gamma function

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$$

applied to $z = 1/2 + x$ becomes

$$\Gamma(1/2 - x)\Gamma(1/2 + x) = \frac{\pi}{\cos(\pi x)}.$$

By repeated applications of $\Gamma(r+1) = r\Gamma(r)$ this yields

$$\Gamma(7/2 - x)\Gamma(7/2 + x) = ((1/2)^2 - x^2)((3/2)^2 - x^2)((5/2)^2 - x^2) \frac{\pi}{\cos(\pi x)}$$

which for $x = \sqrt{5}/2$ reduces to

$$\Gamma\left(\frac{7}{2} - \frac{\sqrt{5}}{2}\right)\Gamma\left(\frac{7}{2} + \frac{\sqrt{5}}{2}\right) = \frac{-5\pi}{\cos(\pi\sqrt{5}/2)}.$$

Substituting this expression in (15) and evaluating the limit with Stirling's formula $\Gamma(n+r) \sim (n/e)^n n^{r-1/2} \sqrt{2\pi}$ yields (6). \square

3. ESTIMATES OF FOURIER TRANSFORMS

Recall that the random Dirichlet series $S = \sum_{n=1}^{\infty} I_n/n^s$ with parameters $s > 0$ and $\beta > 0$ is defined by an independent sequence of Bernoulli random variables I_n taking value 1 with probability $1/n^\beta$ and 0 otherwise. We assume that $s + \beta > 1$ for the almost sure convergence. Let μ be the distribution of S . Here we start with an estimate of the Fourier transform of μ ,

$$\hat{\mu}(t) := \int_0^\infty e^{-2\pi i t x} d\mu(x) \quad (-\infty < t < \infty).$$

Theorem 3.1. *Let $s > 0$ and $0 < \beta \leq 1$ with $s + \beta > 1$.*

- (1) *Let $s > 0$ arbitrary and $\beta = 1$. Then for every small enough $\varepsilon > 0$ there exists a constant $C_{\varepsilon,s} > 0$ such that for every t ,*

$$|\hat{\mu}(t)| \leq C_{\varepsilon,s} |t|^{-\frac{1}{s} + \varepsilon}.$$

In particular, for $0 < s < 2$ the distribution μ has a density in L^2 , and for $0 < s < 1$ it has a bounded continuous density.

- (2) Let $s > 0$ arbitrary and $0 < \beta < 1$ with $s + \beta > 1$. Then there exist constants $C_{\beta,s} > 0$ and $T > 0$ such that for every $|t| \geq T$,

$$|\hat{\mu}(t)| \leq \exp \left(-C_{\beta,s} |t|^{\frac{1-\beta}{s+1}} \right).$$

In particular, the distribution μ has a smooth density.

For the Fourier transform of μ , we have

$$\begin{aligned} |\hat{\mu}(t)|^2 &= \int_{\mathbb{R}^2} e^{-2\pi i t(x-y)} d\mu(x) d\mu(y) \\ &= \prod_{n=1}^{\infty} \left(1 - \frac{2}{n^\beta} + \frac{2}{n^{2\beta}} + 2 \left(\frac{1}{n^\beta} - \frac{1}{n^{2\beta}} \right) \cos \frac{2\pi t}{n^s} \right). \end{aligned}$$

Since $|\hat{\mu}(t)|^2$ is even in t , it is enough to estimate for $t > 0$. To prove Theorem 3.1 (1), we will show that for every $\varepsilon > 0$, there exists an interval $I_t = (a(t), b(t)]$ such that the above product which is restricted to I_t has the desired bound. It is realised by taking I_t as $[t^{\frac{1}{q+2+s}}, t^{\frac{1}{s}}]$, where q is a large enough integer. To prove Theorem 3.1 (2), we will find an interval I_t such that the above product which is restricted to I_t decays sub-exponentially fast. The interval I_t is chosen as $[t^{\frac{1}{s+2}}, t^{\frac{1}{s+1}}]$.

We begin with a lemma which involves an estimate of exponential sums.

Lemma 3.2. Fix $s > 0$. For an integer $q \geq 0$, suppose that f has $q + 2$ continuous derivatives on $(1, \infty)$ such that for some $t \geq 1$

$$|f^{(r)}(x)| \asymp \frac{t}{x^{r+s}}$$

for $r = 1, \dots, q + 2$. Then we have the following:

- (1) For $\beta = 1$, define $\delta_q = \frac{1}{q+2+s}$, $\Delta = \frac{1}{s}$, and the interval $I_{q,s,t} := (2^{\lfloor \log_2 t^{\delta_q} \rfloor}, 2^{\lfloor \log_2 t^\Delta \rfloor}]$. There exists a constant $C_{q,s} > 0$ depending on q and s such that

$$\left| \sum_{n \in I_{q,s,t}} \left(\frac{1}{n^\beta} - \frac{1}{n^{2\beta}} \right) \cos 2\pi f(n) \right| \leq C_{q,s}.$$

- (2) For $0 < \beta < 1$, define $\delta_0 = \frac{1}{s+2}$, $\Delta = \frac{1}{s+1}$, and the interval $I_+ = (2^{\lfloor \log_2 t^{\delta_0} \rfloor}, 2^{\lfloor \log_2 t^\Delta \rfloor}]$. There exists a constant $C_{\beta,s} > 0$ depending on β and s such that

$$\left| \sum_{n \in I_+} \left(\frac{1}{n^\beta} - \frac{1}{n^{2\beta}} \right) \cos 2\pi f(n) \right| \leq C_{\beta,s} t^{\frac{1-\beta}{s+2}}.$$

Here $\lfloor a \rfloor$ denotes the integer part of a .

We employ the following theorem in [GK] to show the above Lemma 3.2. This is the iterated version of [KN, Theorem 2.7, Chapter 1].

Theorem 3.3 ([GK], Theorem 2.9). *Let $q \geq 0$ be an integer. Suppose that f has $q + 2$ continuous derivatives on an interval $I \subset (N, 2N]$. Suppose also that there is a constant F such that*

$$|f^{(r)}(x)| \asymp FN^{-r}, \quad x \in I,$$

for $r = 1, \dots, q + 2$. Then

$$\left| \sum_{n \in I} e^{2\pi i f(n)} \right| \leq c \left(F^{\frac{1}{4Q-2}} N^{1-\frac{q+2}{4Q-2}} + F^{-1} N \right),$$

where the n runs over integers in I in the above summation, $Q = 2^q$ and c is an absolute constant.

Proof of Lemma 3.2. Let $t \geq 1$ be a constant appearing in the order of magnitude of $|f^{(r)}(x)|$. For $0 < \delta_q < \Delta$, consider the interval $I_q = (2^{\lfloor \log_2 t^{\delta_q} \rfloor}, 2^{\lfloor \log_2 t^\Delta \rfloor}]$. Divide the interval I_q as

$$I_q = \bigsqcup_{k=m_q}^{M-1} J_k$$

where

$$\begin{aligned} m_q &= \lfloor \log_2 t^{\delta_q} \rfloor, \quad M = \lfloor \log_2 t^\Delta \rfloor, \\ J_k &= (2^k, 2^{k+1}], \quad m_q \leq k \leq M-1. \end{aligned}$$

Applying Theorem 3.3 on each $I \subset J_k = (2^k, 2^{k+1}]$ with $N = 2^k$,

$$|f^{(r)}(x)| \asymp \frac{t}{N^{r+s}}, \quad F = \frac{t}{N^s},$$

for $r = 1, \dots, q + 2$, we have

$$\left| \sum_{n \in I} e^{2\pi i f(n)} \right| \leq ct^{\frac{1}{4Q-2}} N^{-\frac{s}{4Q-2}} N^{1-\frac{q+2}{4Q-2}} + ct^{-1} N^{1+s}. \quad (16)$$

By summation by parts on each J_k ,

$$\sum_{n \in J_k} \left(\frac{1}{n^\beta} - \frac{1}{n^{2\beta}} \right) \cos 2\pi f(n) = \sum_{l=2^k+1}^{\infty} a_l \sum_{n=2^k+1}^{l \wedge 2^{k+1}} \cos 2\pi f(n),$$

where $a_l = \left(\frac{1}{l^\beta} - \frac{1}{l^{2\beta}} \right) - \left(\frac{1}{(l+1)^\beta} - \frac{1}{(l+1)^{2\beta}} \right)$, and $x \wedge y = \min(x, y)$. Taking $I = (2^k, l \wedge 2^{k+1}] \subset (N, 2N]$, we have by (16) that

$$\begin{aligned} \left| \sum_{n \in J_k} \left(\frac{1}{n^\beta} - \frac{1}{n^{2\beta}} \right) \cos 2\pi f(n) \right| &\leq c_1 N^{-\beta} \left(t^{\frac{1}{4Q-2}} N^{1-\frac{q+2+s}{4Q-2}} + t^{-1} N^{1+s} \right) \\ &= c_1 t^{\frac{1}{4Q-2}} N^{1-\beta-\frac{q+2+s}{4Q-2}} + c_1 t^{-1} N^{1+s-\beta} \\ &= c_1 t^{\frac{1}{4Q-2}} 2^{-k(-1+\beta+\frac{q+2+s}{4Q-2})} + c_1 t^{-1} 2^{k(1+s-\beta)}. \end{aligned}$$

Here, note that the exponents in the two “ $2^{\pm k}$ ”’s are positive:

$$-1 + \beta + \frac{q+2+s}{4Q-2} = \begin{cases} \frac{q+2+s}{4Q-2} > 0, & \text{when (1) } \beta = 1 \text{ and } q \geq 0, \\ \frac{2\beta+s}{2} > 0, & \text{when (2) } 0 < \beta < 1 \text{ and } q = 0, \end{cases}$$

and $1 + s - \beta > 0$ for both cases (1) and (2).

Therefore, for every $t \geq 1$, with $2^{-m_q} \asymp t^{-\delta_q}$ and $2^M \asymp t^\Delta$,

$$\begin{aligned} \sum_{k=m_q}^{M-1} \left| \sum_{n \in J_k} \left(\frac{1}{n^\beta} - \frac{1}{n^{2\beta}} \right) \cos 2\pi f(n) \right| &\leq c_2 t^{\frac{1}{4Q-2}} 2^{-m_q(-1+\beta+\frac{q+2+s}{4Q-2})} + c_2 t^{-1} 2^{M(1+s-\beta)} \\ &\leq c_3 t^{\frac{1}{4Q-2}} t^{-\delta_q(-1+\beta+\frac{q+2+s}{4Q-2})} + c_3 t^{-1} t^{\Delta(1+s-\beta)}, \end{aligned} \quad (17)$$

where the constant c_3 depends on q , β and s .

First, we show (1). Let $\beta = 1$, $\delta_q = \frac{1}{q+2+s}$ and $\Delta = \frac{1}{s}$. Then,

$$\frac{1}{4Q-2} - \delta_q \left(-1 + \beta + \frac{q+2+s}{4Q-2} \right) = 0,$$

and $-1 + \Delta(1+s-\beta) = 0$. Hence, the last sum in (17) is bounded from above by some constant $C_{q,s}$.

Next, we show (2). Let $0 < \beta < 1$, $\delta_0 = \frac{1}{s+2}$ and $\Delta = \frac{1}{s+1}$. When $q = 0$, it follows that

$$\frac{1}{4Q-2} - \delta_0 \left(-1 + \beta + \frac{q+2+s}{4Q-2} \right) = \frac{1-\beta}{s+2} > 0,$$

and $-1 + \Delta(1+s-\beta) = \frac{-\beta}{s+1} < 0$. Then we have the desired bound $C_{\beta,s} t^{\frac{1-\beta}{s+2}}$. \square

Proof of Theorem 3.1. We apply Lemma 3.2 with $f(x) = \frac{t}{x^s}$. First, we prove (1). For all $\varepsilon > 0$ such that $\varepsilon < \frac{1}{s}$, choose an integer q satisfying that $\delta_q = \frac{1}{q+2+s} < \varepsilon$. Then $\Delta - \delta_q > \frac{1}{s} - \varepsilon > 0$. By Lemma 3.2 (1),

$$\begin{aligned} \log |\hat{\mu}(t)|^2 &\leq \sum_{n \in I_q} \left(-\frac{2}{n} + \frac{2}{n^2} + 2 \left(\frac{1}{n} - \frac{1}{n^2} \right) \cos \frac{2\pi t}{n^s} \right) \\ &\leq -2(\Delta - \delta_q) \log t + C_{\varepsilon,s}, \end{aligned}$$

and we obtain the bound $(-\frac{2}{s} + 2\varepsilon) \log t + C_{\varepsilon,s}$ for every $t \geq 1$; hence we have $|\hat{\mu}(t)| \leq C_{\varepsilon,s} t^{-\frac{1}{s} + \varepsilon}$. For $0 < s < 2$ and $\varepsilon < \frac{1}{s} - \frac{1}{2}$, since $\hat{\mu}$ is in L^2 , the distribution has a density in L^2 by the Plancherel theorem [Kat, Theorem 3.1, Chapter VI]. For $0 < s < 1$ and $\varepsilon < \frac{1}{s} - 1$, since $\hat{\mu}$ is in L^1 , the distribution μ has a bounded and continuous density by the Fourier inversion formula.

Next, we show (2). By Lemma 3.2 (2),

$$\begin{aligned} \log |\hat{\mu}(t)|^2 &\leq \sum_{n \in I_+} \left(-\frac{2}{n^\beta} + \frac{2}{n^{2\beta}} + 2 \left(\frac{1}{n^\beta} - \frac{1}{n^{2\beta}} \right) \cos \frac{2\pi t}{n^s} \right) \\ &\leq \sum_{n \in I_+} -\frac{2}{n^\beta} \left(1 - \frac{1}{n^\beta} \right) + C_{\beta,s} t^{\frac{1-\beta}{s+2}}. \end{aligned}$$

Since $\beta < 1$, the first term in the last line is bounded by $-C_\beta t^{\Delta(1-\beta)}$. Here $\Delta(1-\beta) = \frac{1-\beta}{s+1}$ in $(0, 1)$, and this is greater than $\frac{1-\beta}{s+2}$; hence there exists T such that for every $t \geq T$, we have the bound $-C_{\beta,s}|t|^{\frac{1-\beta}{s+1}}$. Since $\hat{\mu}$ decays faster than any polynomial, μ has a smooth density. \square

Proof of Corollary 1.2. The decay of the Fourier transform implies that the density of μ is in C^r for $0 < s < 1/(r+1)$ and an integer $r \geq 0$. We note that $\hat{\mu}$ is in L^p for every $p > s$ by Theorem 3.1(1). For $1 \leq s < 2$, by the Hausdorff-Young inequality [Kat, Theorem 3.2, Chapter VI], the density f of μ satisfies that $\|f\|_{L^q} \leq \|\hat{f}\|_{L^p}$, for every $s < p \leq 2$, and the conjugate q of p . Therefore the density of μ is in L^q for every $2 \leq q < s/(s-1)$. The density of μ is a priori in L^1 ; by the inequality $\|f\|_{L^r} \leq \|f\|_{L^p}^\theta \|f\|_{L^q}^{1-\theta}$ for $1 \leq p < r < q \leq \infty$, where $\theta = \left(\frac{1}{r} - \frac{1}{q}\right) / \left(\frac{1}{p} - \frac{1}{q}\right)$, we see the density of μ is in L^q for every $1 \leq q < s/(s-1)$ as well. \square

Theorem 2.2 follows from the same argument as Theorem 3.1 and Corollary 1.2.

Proof of Theorem 2.2. Fix $s = 1$ and apply to Lemma 3.2 (1) for $f(x) = -t \log(1 - \frac{1}{x})$. Note that on $[2, \infty)$, we have $|f^{(r)}| \asymp \frac{t}{x^{r+1}}$ for $0 \leq r \leq q+2$. It follows that the Fourier transform of the distribution μ_∞ of S_∞ satisfies the same estimate as the one for μ in the case when $\beta = 1$ and $s = 1$ in Theorem 3.1(1). Then, as in Corollary 1.2, the distribution μ_∞ has a density in L^q for all $1 \leq q < \infty$. Since $V_\infty = e^{-S_\infty}$, the distribution of V_∞ is absolutely continuous with respect to Lebesgue measure. \square

4. COMPLETION OF THE PROOF OF THEOREM 1.1.

We consider the case when $s > 0$ and $\beta = 1$, namely, $S := \sum_{n=1}^\infty I_n/n^s$, where the I_n are independent and $I_n = 1$ with probability $1/n$ and $I_n = 0$ otherwise.

Theorem 4.1. *For every $s > 0$ and $\beta = 1$, the distribution μ of S is absolutely continuous with respect to Lebesgue measure.*

Proof. Fix $s > 0$. It suffices to show that there is a sequence (A_m) of events with $\mathbb{P}(A_m) \rightarrow 1$ as $m \rightarrow \infty$, such that conditioned on A_m , the distribution of S is absolutely continuous. Indeed, this follows from the formula,

$$\mathbb{P}(S \in B|A) = \frac{\mathbb{P}(\{S \in B\} \cap A)}{\mathbb{P}(A)} \geq \frac{\mathbb{P}(S \in B) - (1 - \mathbb{P}(A))}{\mathbb{P}(A)},$$

since if there is a set B of Lebesgue measure 0 with $\mathbb{P}(S \in B) > 0$ then also $\mathbb{P}(S \in B|A) > 0$ when $\mathbb{P}(A)$ is sufficiently close to 1.

Rewrite the series as

$$S = 1 + \sum_{k=0}^{\infty} Y_k,$$

where Y_k is the portion of the sum taken on the k -th block $(2^k, 2^{k+1}]$, i.e.,

$$Y_k = \sum_{n=2^k+1}^{2^{k+1}} \frac{I_n}{n^s}.$$

Define a sequence of independent Bernoulli random variables (M_k) by setting $M_k = 1$ if and only if $I_n = 1$ for exactly one n in the k -th block. It is straightforward to check that there exists some $p > 0$ such that $\mathbb{P}(M_k = 1) \geq p$ for all k (e.g. $p = \frac{1}{2} \log 2$). Thus the sequence (M_k) dominates an i.i.d. sequence of Bernoulli random variables with parameter p . We proceed to define the events A_m . Define a large constant a by $a = 2/\log(1/(1-p))$. Define a sequence of events (B_m) by

$$B_m = \bigcup_{m \leq k \leq m + \lfloor a \log m \rfloor} \{M_k = 1\}.$$

Define $A_m = \bigcap_{l \geq m} B_l$. To check that $\mathbb{P}(A_m) \rightarrow 1$ as $m \rightarrow \infty$, we need only notice that the choice of a ensures that $\mathbb{P}(B_l^c) \leq 1/l^2$ and we have $\mathbb{P}(A_m^c) \leq \sum_{l=m}^{\infty} \mathbb{P}(B_l^c)$, where A^c denotes the complement of A .

From now on fix m and condition on the entire sequence (M_k) , at a sample point where the event A_m is satisfied. Observe that the random variables (Y_k) are still independent under this conditioning. For a k such that $M_k = 1$ the remaining randomness in Y_k is exactly which n is the single n in the k -th block for which $I_n = 1$. This n is distributed in the block according to the probabilities $1/(z_k(n-1))$, where z_k is a normalising constant which tends to $\log 2$ as k grows, and $z_k \geq \log 2$.

Write $\hat{\mu}_M(t)$ for the Fourier transform of the distribution of S conditioned on a sample sequence $M = (M_k)$, and $\hat{\mu}_k(t)$ for the Fourier transform of the distribution of Y_k conditioned on M_k . Here we define $\hat{\mu}_M(t)$ as the Fourier transform of the regular conditional probability given $M = (M_k)$. (See e.g., [B, Chapter 4.3] on the existence of a regular conditional probability.) The Fourier transform of the distribution of S conditioned on A_m is given by

$$\frac{1}{\mathbb{P}(A_m)} \int_{A_m} \hat{\mu}_{M(\omega)}(t) \mathbb{P}(d\omega). \quad (18)$$

We have $\hat{\mu}_M(t) = e^{2\pi i t} \prod_{k=0}^{\infty} \hat{\mu}_k(t)$ for \mathbb{P} -almost every sequence $M = (M_k)$, and $|\hat{\mu}_k(t)| \leq 1$ for all k and t . If k is such that $M_k = 1$, then we have

$$\hat{\mu}_k(t) = \sum_{n=2^{k+1}}^{2^{k+1}} \frac{1}{z_k(n-1)} \exp \frac{2\pi i t}{n^s}.$$

Apply Theorem 3.3 in the case where $q = 0$, $N = 2^k$, $f(x) = t/x^s$ and $F = t/N^s$, then for $I \subset (N, 2N]$, we have

$$\left| \sum_{n \in I} e^{2\pi i f(n)} \right| \leq c \left(t^{\frac{1}{2}} N^{-\frac{s}{2}} + t^{-1} N^{1+s} \right),$$

where the constant c is absolute. By summation by parts on $(N, 2N]$ as in the proof of Lemma 3.2, using $z_k \geq \log 2$, we have

$$\begin{aligned} |\hat{\mu}_k(t)| &= \left| \sum_{n=N+1}^{2N} \frac{1}{z_k(n-1)} \exp \frac{2\pi i t}{n^s} \right| \leq c N^{-1} \left(t^{\frac{1}{2}} N^{-\frac{s}{2}} + t^{-1} N^{1+s} \right) \\ &\leq c t^{\frac{1}{2}} 2^{-k(\frac{s}{2}+1)} + c t^{-1} 2^{ks}. \end{aligned}$$

Fix some σ_1 and σ_2 such that $s < \sigma_1 < \sigma_2 < s+2$ (e.g., $\sigma_1 = s+0.1$ and $\sigma_2 = s+1.9$). Then for k such that $\frac{1}{\sigma_2} \log_2 t < k < \frac{1}{\sigma_1} \log_2 t$, we have

$$\frac{1}{2} \log_2 t - k \left(\frac{s}{2} + 1 \right) < -\frac{s+2-\sigma_2}{2\sigma_2} \log_2 t = -\delta_2 \log_2 t$$

and

$$-\log_2 t + ks < -\left(1 - \frac{s}{\sigma_1}\right) \log_2 t = -\delta_1 \log_2 t,$$

where δ_1 and δ_2 are positive and depend only on s , σ_1 and σ_2 . Define $\delta := \min(\delta_1, \delta_2) > 0$. To summarize, if k is such that $M_k = 1$ and $\frac{1}{\sigma_2} \log_2 t < k < \frac{1}{\sigma_1} \log_2 t$, then we have

$$|\hat{\mu}_k(t)| \leq ct^{-\delta} \quad (19)$$

for every $t \geq 1$, where the constant c is absolute.

By the definition of A_m , the number of k in the interval $\left[\frac{1}{\sigma_2} \log_2 t, \frac{1}{\sigma_1} \log_2 t\right]$ having $M_k = 1$ tends to infinity as t tends to infinity. Since (19) holds for all k in this interval having $M_k = 1$ and since there are more and more of these as t grows, we conclude that $\hat{\mu}_M(t)$, as a product of all $\hat{\mu}_k(t)$, decays faster than $(c|t|^{-\delta})^L$ in t for all $L > 0$. This decay of $\hat{\mu}_M(t)$ holds uniformly on A_m modulo \mathbb{P} -measure null set. Therefore (18) decays faster than polynomially in t . This proves that conditioned on A_m , the distribution of S is smooth, in particular, absolutely continuous, and concludes the proof. \square

Theorem 4.2. *For every $s > 1$ and $\beta = 1$, the density of μ is unbounded on every interval in its support.*

Proof. Let $D_N = \{\sum_{n=1}^N \varepsilon_n/n^s \mid \varepsilon_n = 0 \text{ or } 1\}$ be the set of all possible values of the sums up to the N -th term in the series S . Note that $\bigcup_N D_N$ is dense in the support of μ . For every $M > N$, we write the series as the sum of three independent random variables: $S = S_1 + S_2 + S_3$, where

$$S_1 := \sum_{n=1}^N I_n/n^s, \quad S_2 := \sum_{n=N+1}^M I_n/n^s, \quad \text{and} \quad S_3 := \sum_{n=M+1}^{\infty} I_n/n^s.$$

We have $\mathbb{P}(S_1 = x) = c_x > 0$ for each $x \in D_N$, and $\mathbb{P}(S_2 = 0) = \prod_{n=N+1}^M (1 - 1/n) \geq C/M$. Since $\mathbb{E}S_3 \leq C/M^s$, we have $\mathbb{P}(S_3 \leq 2C/M^s) \geq 1/2$ by Markov's inequality. Therefore we obtain $\mathbb{P}(S \in [x, x + 2C/M^s]) \geq c'_x/M$ for some constant $c'_x > 0$ depending only on x and for arbitrary large M . Thus, for every $s > 1$, the density is unbounded on every interval in its support. \square

Remark 4.3. We see the sharpness of the estimate of Fourier transform $\hat{\mu}(t)$ in part 1 of Theorem 1.1 for $1 < s < 2$. Since μ has a density in L^q for all $2 \leq q < s/(s-1)$ by Corollary 1.2, Hölder's inequality gives that $\mathbb{P}(S \in [x, x + \varepsilon]) = o(\varepsilon^{1/p})$, as $\varepsilon \rightarrow 0$, for every $s < p \leq 2$. This is sharp. Indeed, in the above argument, for each x in the union of the D_N , and for $\varepsilon = 2C/M^s$, we have $\mathbb{P}(S \in [x, x + \varepsilon]) \geq c_x \varepsilon^{1/s}$.

5. SINGULARITY FOR THE PRIME NUMBERS SEQUENCE.

Proof of Theorem 1.4. Let $S := S_{primes}$. Fix $0 < \varepsilon < s$. Decompose the series into three parts: $S = S_1 + S_2 + S_3$, where

$$S_1 := \sum_{p \leq N^\varepsilon} \frac{I_p}{p^s}, \quad S_2 := \sum_{N^\varepsilon < p < N} \frac{I_p}{p^s}, \quad \text{and} \quad S_3 := \sum_{N \leq p} \frac{I_p}{p^s}.$$

Since $\mathbb{E}S_3 = \sum_{N \leq p} 1/p^{1+s} \leq C/N^s$ for some constant C , we have $\mathbb{P}(S_3 \geq 2C/N^s) \leq 1/2$ by Markov's inequality. Hence $\mathbb{P}(S_3 \leq 2C/N^s) \geq 1/2$.

Let us show that the probability that $S_2 = 0$ is bounded away from below by a positive constant independent of N . We use Mertens' theorem,

$$\prod_{p < N} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma+o(1)}}{\log N}, \quad (20)$$

where γ is the Euler-Mascheroni constant ([HW, Theorem 429] and [M]). Then we have

$$\mathbb{P}(S_2 = 0) = \prod_{N^\varepsilon < p < N} \left(1 - \frac{1}{p}\right) = \varepsilon e^{o(1)} \geq \varepsilon c_1,$$

where c_1 is absolute.

Fix a positive integer $m < N^\varepsilon$. Define the sequence $\{a_p(m)\}_{p < N^\varepsilon}$ such that $a_p(m)$ is 1 if $p|m$, and 0 otherwise. Let $x_m := \sum_{p < N^\varepsilon} a_p(m)/p^s$. We claim that the probability that S_1 coincides with some x_m is bounded away from 0 by a constant independent of N . We consider the event $A_m := \{I_p = a_p(m) \text{ for all } p < N^\varepsilon\}$. Notice that the A_m are not disjoint (e.g., $A_2 = A_4$), but A_m for square-free m 's are disjoint. Then,

$$\begin{aligned} \mathbb{P}(S_1 = x_m \text{ for some } m < N^\varepsilon) &\geq \mathbb{P}\left(\bigcup_{m < N^\varepsilon} A_m\right) \\ &= \sum_{m < N^\varepsilon, m \text{ is square-free}} \mathbb{P}(A_m) \\ &= \sum_{m < N^\varepsilon} |\mu(m)| \prod_{p|m} \frac{1}{p-1} \prod_{p < N^\varepsilon} \left(1 - \frac{1}{p}\right), \end{aligned}$$

where $\mu(m)$ is the Möbius function, i.e.,

- (i) $\mu(1) = 1$,
- (ii) $\mu(m) = 0$ if m has a squared factor,
- (iii) $\mu(p_1 p_2 \cdots p_k) = (-1)^k$ if all the primes p_1, p_2, \dots, p_k are different.

The last sum is bounded from below by

$$\sum_{m < N^\varepsilon} \frac{|\mu(m)|}{m} \prod_{p < N^\varepsilon} \left(1 - \frac{1}{p}\right) \quad (21)$$

since $\prod_{p|m} p \leq m$. Note that the number of square-free numbers up to x grows linearly in x ; more precisely,

$$\sum_{m \leq x} |\mu(m)| = \frac{6}{\pi^2} x + O(\sqrt{x})$$

by [HW, Theorem 334]. By summation by parts, we have that

$$\sum_{m \leq N^\varepsilon} \frac{|\mu(m)|}{m} = \frac{6}{\pi^2} (1 + o(1)) \log N^\varepsilon.$$

This and Mertens' theorem (20) imply that the sum (21) is at least some positive constant c_2 which is independent of N and ε .

Let us define the set B_N as a union of intervals $[x_m, x_m + 2C/N^s]$ for $m < N^\varepsilon$. Since S_1 , S_2 and S_3 are independent, we have $\mathbb{P}(S \in B_N) \geq c_1 c_2 \varepsilon / 2$. On the other hand, the Lebesgue measure of B_N is at most $2CN^{\varepsilon-s}$, which tends to 0 as $N \rightarrow \infty$. Hence the distribution cannot be absolutely continuous. Recall that the distribution is continuous if and only if

$$\sum_p \left(1 - \sup_{x \in \mathbb{R}} \mathbb{P} \left(\frac{I_p}{p^s} = x \right) \right) = \infty$$

[Ell, Lemma 1.22]. It is satisfied since $\sum_p 1/p = \infty$ (e.g., [HW, Theorem 427]). The Jessen-Wintner law of pure type [JW] implies that the distribution is non-atomic singular. \square

Remark 5.1. By a straight forward adaptation to the above proof, Theorem 1.4 is generalized as follows: Let $(a_p)_{p;\text{primes}}$ be a sequence of real numbers such that

- (i) $\sum_{p, a_p \neq 0} \frac{1}{p} = \infty$,
- (ii) $\sum_p \frac{|a_p|}{p} < \infty$,
- (iii) $\sum_{p > x} \frac{|a_p|}{p} = o(x^{-c})$ for some $c > 0$,

and (I_p) be an independent sequence with value 0 or 1 with probability $1 - 1/p$ or $1/p$, respectively. Then $S = \sum_p a_p I_p$ converges almost surely, and its distribution is non-atomic singular.

6. FURTHER QUESTIONS

Below we list some natural questions about the critical case when $s = 1$ and $\beta = 1$. Let $\beta = 1$. By part 1 of Theorem 1.1, for $0 < s < 1$ the distribution μ has a bounded continuous density, while for $s > 1$ it has an unbounded density. In the case when $s = 1$, the density of μ is in L^q for every $1 \leq q < \infty$ by Corollary 1.2. In fact, we conjecture the following.

Conjecture 6.1. *For $s = 1$ and $\beta = 1$, the density is bounded.*

We also ask the following question about the critical case.

Question 6.2. *For $s = 1$ and $\beta = 1$, is the density discontinuous?*

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